5. False. Consider $\lim_{x\to 5}\dfrac{x(x-5)}{x-5}$ or $\lim_{x\to 5}\dfrac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

6. False. Consider $\lim_{x\to 6}[f(x)g(x)] = \lim_{x\to 6}\left[(x-6)\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6)=0$ and $g(6)$ does not exist, so $f(6)g(6)\neq 1$.

7. True. A polynomial is continuous everywhere, so $\lim_{x\to b} p(x)$ exists and is equal to $p(b)$.

8. False. Consider $\lim_{x\to 0}[f(x)-g(x)] = \lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

9. True. See Figure 4 in Section 2.6.

10. False. Consider $f(x)=\sin x$ for $x\geq 0$. $\lim_{x\to\infty} f(x)\neq\pm\infty$ and f has no horizontal asymptote.

11. False. Consider $f(x)=\begin{cases}1/(x-1) & \text{if } x\neq 1\\ 2 & \text{if } x=1\end{cases}$

12. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let

$f(x)=\begin{cases}1 & \text{if } 0\leq x<3\\ -1 & \text{if } x=3\end{cases}$ There is no number $c\in[0,3]$ with $f(c)=0$.

13. True. Use Theorem 2.5.8 with $a=2$, $b=5$, and $g(x)=4x^2-11$. Note that $f(4)=3$ is not needed.

14. True. Use the Intermediate Value Theorem with $a=-1$, $b=1$, and $N=\pi$, since $3<\pi<4$.

15. True, by the definition of a limit with $\varepsilon=1$.

16. False. For example, let $f(x)=\begin{cases}x^2+1 & \text{if } x\neq 0\\ 2 & \text{if } x=0\end{cases}$

Then $f(x)>1$ for all x, but $\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(x^2+1\right)=1$.

17. False. See the note after Theorem 4 in Section 2.9.

18. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x\to r} f(x)=f(r)$.

EXERCISES

1. (a) (i) $\lim_{x\to 2^+} f(x)=3$

(ii) $\lim_{x\to -3^+} f(x)=0$

(iii) $\lim_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x\to 4} f(x)=2$

(v) $\lim_{x\to 0} f(x)=\infty$

(vi) $\lim_{x\to 2^-} f(x)=-\infty$

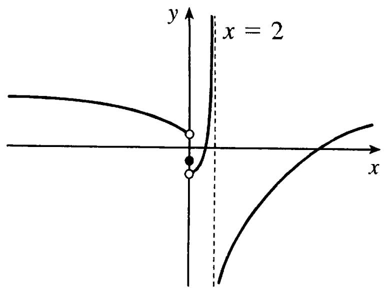
(vii) $\lim_{x\to\infty} f(x)=4$

(viii) $\lim_{x\to -\infty} f(x)=-1$

(b) The equations of the horizontal asymptotes are $y=-1$ and $y=4$.

(c) The equations of the vertical asymptotes are $x=0$ and $x=2$.

(d) f is discontinuous at $x=-3, 0, 2$, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2. 

3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1.$

4. Since rational functions are continuous, $\lim_{x\to 3}\dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0.$

5. $\lim_{x\to -3}\dfrac{x^2-9}{x^2+2x-3} = \lim_{x\to -3}\dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3}\dfrac{x-3}{x-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim_{x\to 1^+}\dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3\to 0$ as $x\to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $\lim_{h\to 0}\dfrac{(h-1)^3+1}{h} = \lim_{h\to 0}\dfrac{(h^3-3h^2+3h-1)+1}{h} = \lim_{h\to 0}\dfrac{h^3-3h^2+3h}{h} = \lim_{h\to 0}(h^2-3h+3) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\lim_{h\to 0}\frac{(h-1)^3+1}{h} = \lim_{h\to 0}\frac{(h-1)^3+1^3}{h} = \lim_{h\to 0}\frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}$$
$$= \lim_{h\to 0}\left[(h-1)^2-h+2\right] = 1-0+2 = 3$$

8. $\lim_{t\to 2}\dfrac{t^2-4}{t^3-8} = \lim_{t\to 2}\dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2}\dfrac{t+2}{t^2+2t+4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim_{r\to 9}\dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4\to 0$ as $r\to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r\neq 9$.

10. $\lim_{v\to 4^+}\dfrac{4-v}{|4-v|} = \lim_{v\to 4^+}\dfrac{4-v}{-(4-v)} = \lim_{v\to 4^+}\dfrac{1}{-1} = -1$

11. $\lim_{s\to 16}\dfrac{4-\sqrt{s}}{s-16} = \lim_{s\to 16}\dfrac{4-\sqrt{s}}{(\sqrt{s}+4)(\sqrt{s}-4)} = \lim_{s\to 16}\dfrac{-1}{\sqrt{s}+4} = \dfrac{-1}{\sqrt{16}+4} = -\dfrac{1}{8}$

12. $\lim_{v\to 2}\dfrac{v^2+2v-8}{v^4-16} = \lim_{v\to 2}\dfrac{(v+4)(v-2)}{(v+2)(v-2)(v^2+4)} = \lim_{v\to 2}\dfrac{v+4}{(v+2)(v^2+4)} = \dfrac{2+4}{(2+2)(2^2+4)} = \dfrac{3}{16}$

13. $$\frac{|x-8|}{x-8} = \begin{cases} \dfrac{x-8}{x-8} & \text{if } x-8>0 \\[2mm] \dfrac{-(x-8)}{x-8} & \text{if } x-8<0 \end{cases} = \begin{cases} 1 & \text{if } x>8 \\ -1 & \text{if } x<8 \end{cases}$$

Thus, $\lim_{x\to 8^-}\dfrac{|x-8|}{x-8} = \lim_{x\to 8^-}(-1) = -1.$

14. $\lim_{x\to 9^+}\left(\sqrt{x-9}+[\![x+1]\!]\right) = \lim_{x\to 9^+}\sqrt{x-9} + \lim_{x\to 9^+}[\![x+1]\!] = \sqrt{9-9}+10 = 10$

15. $\lim_{x\to 0}\dfrac{1-\sqrt{1-x^2}}{x}\cdot\dfrac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim_{x\to 0}\dfrac{1-(1-x^2)}{x(1+\sqrt{1-x^2})} = \lim_{x\to 0}\dfrac{x^2}{x(1+\sqrt{1-x^2})} = \lim_{x\to 0}\dfrac{x}{1+\sqrt{1-x^2}} = 0$

16. $\lim_{x\to 2}\dfrac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)}\cdot\dfrac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim_{x\to 2}\dfrac{-(x-2)}{x(x-2)(\sqrt{x+2}+\sqrt{2x})} = \lim_{x\to 2}\dfrac{-1}{x\left(\sqrt{x+2}+\sqrt{2x}\right)} = -\dfrac{1}{8}$

17. $\displaystyle\lim_{x\to\infty}\frac{1+2x-x^2}{1-x+2x^2}=\lim_{x\to\infty}\frac{(1+2x-x^2)/x^2}{(1-x+2x^2)/x^2}=\lim_{x\to\infty}\frac{1/x^2+2/x-1}{1/x^2-1/x+2}=\frac{0+0-1}{0-0+2}=-\frac{1}{2}$

18. $\displaystyle\lim_{x\to-\infty}\frac{5x^3-x^2+2}{2x^3+x-3}=\lim_{x\to-\infty}\frac{(5x^3-x^2+2)/x^3}{(2x^3+x-3)/x^3}=\lim_{x\to-\infty}\frac{5-1/x+2/x^3}{2+1/x^2-3/x^3}=\frac{5-0+0}{2+0-0}=\frac{5}{2}$

19. Since x is positive, $\sqrt{x^2}=|x|=x$.

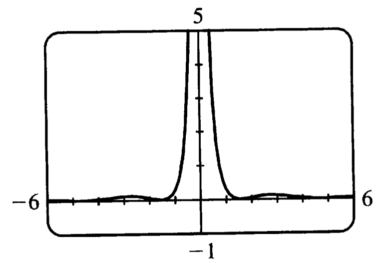
$$\lim_{x\to\infty}\frac{\sqrt{x^2-9}}{2x-6}=\lim_{x\to\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x}=\lim_{x\to\infty}\frac{\sqrt{1-9/x^2}}{2-6/x}=\frac{\sqrt{1-0}}{2-0}=\frac{1}{2}$$

20. $\displaystyle\lim_{x\to10^-}\ln(100-x^2)=-\infty$ since as $x\to10^-$, $(100-x^2)\to0^+$.

21. $\displaystyle\lim_{x\to\infty}e^{-3x}=0$ since $-3x\to-\infty$ as $x\to\infty$ and $\displaystyle\lim_{t\to-\infty}e^t=0$.

22. If $y=x^3-x=x(x^2-1)$, then as $x\to\infty$, $y\to\infty$. $\displaystyle\lim_{x\to\infty}\arctan(x^3-x)=\lim_{y\to\infty}\arctan y=\frac{\pi}{2}$ by (2.6.4).

23. From the graph of $y=(\cos^2 x)/x^2$, it appears that $y=0$ is the horizontal asymptote and $x=0$ is the vertical asymptote. Now $0\le(\cos x)^2\le1$ $\Rightarrow$ $\dfrac{0}{x^2}\le\dfrac{\cos^2 x}{x^2}\le\dfrac{1}{x^2}$ $\Rightarrow$ $0\le\dfrac{\cos^2 x}{x^2}\le\dfrac{1}{x^2}$. But $\displaystyle\lim_{x\to\pm\infty}0=0$ and $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x^2}=0$, so by the Squeeze Theorem,



$\displaystyle\lim_{x\to\pm\infty}\frac{\cos^2 x}{x^2}=0$. Thus, $y=0$ is the horizontal asymptote. $\displaystyle\lim_{x\to0}\frac{\cos^2 x}{x^2}=\infty$ because $\cos^2 x\to1$ and $x^2\to0$ as $x\to0$, so $x=0$ is the vertical asymptote.

24. From the graph of $y=f(x)=\sqrt{x^2+x+1}-\sqrt{x^2-x}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for f, let's multiply and divide it by its conjugate.

$$f_1(x)=\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$
$$=\frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}=\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

Now

$$\lim_{x\to\infty}f_1(x)=\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$
$$=\lim_{x\to\infty}\frac{2+(1/x)}{\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}}\qquad\text{(since }\sqrt{x^2}=x\text{ for }x>0)$$
$$=\frac{2}{1+1}=1,$$

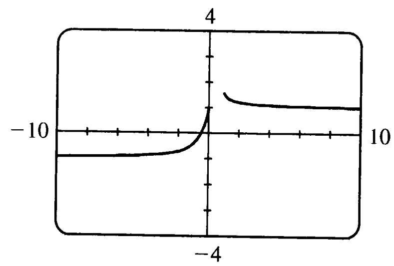
so $y=1$ is a horizontal asymptote. For $x<0$, we have $\sqrt{x^2}=|x|=-x$, so when we divide the denominator by x, with $x<0$, we get

$$\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{x}=-\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2}}=-\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}\right]$$

Therefore,

$$\lim_{x\to-\infty} f_1(x) = \lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$
$$= \lim_{x\to\infty} \frac{2+(1/x)}{-\left[\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}\right]} = \frac{2}{-(1+1)} = -1,$$

so $y=-1$ is a horizontal asymptote. As $x \to 0^-$, $f(x) \to 1$, so $x=0$ is *not* a vertical asymptote. As $x \to 1^+$, $f(x) \to \sqrt{3}$, so $x=1$ is *not* a vertical asymptote and hence there are no vertical asymptotes.



25. Since $2x-1 \le f(x) \le x^2$ for $0<x<3$ and $\lim_{x\to1}(2x-1) = 1 = \lim_{x\to1} x^2$, we have $\lim_{x\to1} f(x) = 1$ by the Squeeze Theorem.

26. Let $f(x) = -x^2$, $g(x) = x^2\cos(1/x^2)$ and $h(x) = x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x \ne 0$, we have $f(x) \le g(x) \le h(x)$ for $x \ne 0$, and so $\lim_{x\to0} f(x) = \lim_{x\to0} h(x) = 0 \Rightarrow \lim_{x\to0} g(x) = 0$ by the Squeeze Theorem.

27. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0<|x-5|<\delta$, then $|(7x-27)-8|<\varepsilon \;\Leftrightarrow\; |7x-35|<\varepsilon \;\Leftrightarrow\; |x-5|<\varepsilon/7$. So take $\delta = \varepsilon/7$. Then $0<|x-5|<\delta \;\Rightarrow\; |(7x-27)-8|<\varepsilon$. Thus, $\lim_{x\to5}(7x-27) = 8$ by the definition of a limit.

28. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0<|x-0|<\delta$, then $|\sqrt[3]{x}-0|<\varepsilon$. Now $|\sqrt[3]{x}-0| = |\sqrt[3]{x}| < \varepsilon \;\Rightarrow\; |x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0<|x-0| = |x| < \varepsilon^3 \;\Rightarrow\; |\sqrt[3]{x}-0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x\to0}\sqrt[3]{x} = 0$.

29. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0<|x-2|<\delta$, then $\left|x^2-3x-(-2)\right|<\varepsilon$. First, note that if $|x-2|<1$, then $-1<x-2<1$, so $0<x-1<2 \;\Rightarrow\; |x-1|<2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0<|x-2|<\delta$ $\Rightarrow\; \left|x^2-3x-(-2)\right| = |(x-2)(x-1)| = |x-2|\,|x-1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim_{x\to2}\left(x^2-3x\right) = -2$ by the definition of a limit.

30. Given $M>0$, we need $\delta>0$ such that if $0<x-4<\delta$, then $2/\sqrt{x-4} > M$. This is true $\Leftrightarrow$ $\sqrt{x-4} < 2/M \;\Leftrightarrow\; x-4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0<x-4<\delta \;\Rightarrow\; 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim_{x\to4^+}\left(2/\sqrt{x-4}\right) = \infty$.

31. (a) $f(x) = \sqrt{-x}$ if $x<0$, $f(x) = 3-x$ if $0 \le x < 3$, $f(x) = (x-3)^2$ if $x>3$.

(i) $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(3-x) = 3$

(ii) $\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim_{x\to0} f(x)$ does not exist.

(iv) $\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(3-x) = 0$

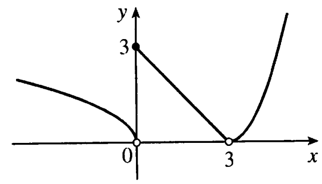
(v) $\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(x-3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{x\to3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim_{x\to 0} f(x)$ does not exist.

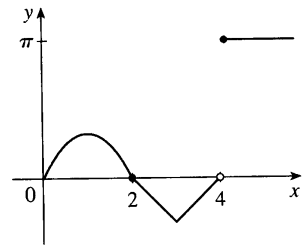
f is discontinuous at 3 since $f(3)$ does not exist.

(c)



32. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2 - x$ if $2 < x \le 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$. Therefore, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (2x - x^2) = 0$ and $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (2 - x) = 0$. Thus, $\lim_{x\to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (2 - x) = -1$ and

(b)



$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} (x - 4) = -1$. Thus, $\lim_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3.
$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x - 4) = 0$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \pi = \pi$. Thus, $\lim_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

33. $\sin x$ is continuous on $\mathbb{R}$ by Theorem 7 in Section 2.5. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 9 in Section 2.5. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 4 in Section 2.5.

34. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \ge 0\} = (-\infty, -3] \cup [3, \infty)$. Note that $x^2 - 2 \ne 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$.

35. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

36. $f(x) = e^{-x^2} - x$ is continuous on $\mathbb{R}$ so it is continuous on $[0, 1]$. $f(0) = 1 > 0 > 1/e - 1 = f(1)$. So by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that $f(c) = 0$. Thus, $e^{-x^2} - x = 0$, or $e^{-x^2} = x$, has a root in $(0, 1)$.

37. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x\to 2} \frac{8 - 2x^2}{x - 2} = \lim_{x\to 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x\to 2} \frac{-2(x - 2)(x + 2)}{x - 2}$$
$$= \lim_{x\to 2} [-2(x + 2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

38. For a general point with x-coordinate a, we have

$$m = \lim_{x\to a} \frac{2/(1 - 3x) - 2/(1 - 3a)}{x - a} = \lim_{x\to a} \frac{2(1 - 3a) - 2(1 - 3x)}{(1 - 3a)(1 - 3x)(x - a)}$$
$$= \lim_{x\to a} \frac{6(x - a)}{(1 - 3a)(1 - 3x)(x - a)} = \lim_{x\to a} \frac{6}{(1 - 3a)(1 - 3x)} = \frac{6}{(1 - 3a)^2}$$